A mathematical model of a guitar string

Rasmus Storjohann*

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A mathematical model of a vibrating guitar string can be used to investigate the effect of pluck point and string properties on the sound of the string, and on how the sound of the string evolves over time from the pluck to when the sound has fully died out. When the string is plucked, it is pulled away from its equilibrium position. Like all physical systems, it will seek to retain its equilibrium position. We have constructed guitars in such a way that this return to the equilibrium position has the side effect of producing a pleasant sound. As the string returns to the equilibrium, it undergoes a number of physical processes:

1 Vibration

As the string moves from an initial non-straight shape to a straight shape, it converts potential energy (which it has due to its non-straight, non-equilibrium state) into kinetic energy. By the time the string is straight, it has so much speed that it overshoots and becomes non-straight in the opposite direction. It will then turn around and move towards a straight shape again, again picking up speed, again overshooting, etc. This is similar to the conversion between kinetic and potential energy in a pendulum. For a string tuned to concert A, this process happens 440 times a second, giving rise to the pitch of the tone. The tighter the string, the faster it will move, and the higher the pitch. The heavier the string, the slower it will move, and the lower the pitch.

2 Harmonics

As described above, the vibrational frequency of a guitar string is determined by the tension and the weight of the string. If you have a string tuned to concert A and try to make it vibrate at the pitch of C below concert A, the string will not do it, any such vibration will instantly die out. However, the guitar string will happily vibrate at frequencies that are any whole number times the frequency of the concert A. This gives rise to the overtone series. So a string tuned to concert A will vibrate at 440 Hz, 880 Hz, 1320 Hz (3×440), etc.

Each such vibrational frequency corresponds to a particular "spatial" frequency along the length of the vibrating string. A string tuned to 440 Hz that is vibrating at 440 Hz will move in such a way that the string describes a single arch (going up then down then up again as the string vibrates), from one end of the string to the other, with stationary "nodes" at the two ends of the string. The same string vibrating at 880 Hz will move so that the string describes two archs (one going up and one going down), with a stationary "node" at the midpoint as well as at the ends. At 1320 Hz, the string will describe three archs, with two nodes at the $\frac{1}{3}$ and $\frac{2}{3}$ points along the string as well as at the end points. The reason that the string will not want to vibrate at other than these very specific frequencies is that in order for it to do so, there would not be nodes at the end points, i.e. the string would have to move at the end points. But since the string is attached at the end points, no such movement is possible, and the string will stubbornly refuse to vibrate at other frequencies than those that fall on the overtone series.

The vibrational movement described above is initialized by the pluck. The pluck forces the string into some shape that it wants to get away from to regain its equilibrium shape. Exactly what the initial shape of the string is, is determined to a large degree by the player: where does she pluck the string, how hard, does she use her fingertip, her nail or a pick, etc, etc. As any player knows, these choices have a large effect on the resulting sound. The reason is that the initial shape of the string at the time of the pluck determines the initial strengths of the different overtones of the sound.

The mathematical expression below includes where the string is being plucked, i.e. close to the middle of the string or close to the end. The other aspects of playing technique (figertip vs. nail vs. pick, etc.) are way to complex for me to handle. However, with the expression below, it is possible to compute the initial intensity of all the harmonics (or partials) based on where the string is plucked.

3 Decay

After the string is released, it vibrates as mentioned above. However, as time passes, the vibration dies out (decays). As the sound decays, it also changes in quality. In particular, the sound of the guitar string as two distinct phases, known as the attack and decay, with quite distinct qualities to the sound. The reason for this is that the different harmonics each decay at different rates. The decay is caused by a number of different mechanisms, that together drain the vibrating string of its energy, allowing it finally (when the sound ceases) to return to its equilibrium position. The mechanisms of decay are

1. Resistance to bending of the string
2. Air resistance braking the movement of the string
3. Transfer of energy from the string to the body of the guitar

There is a very good paper by V.E. Howle and Lloyd N. Trefethen called "Eigenvalues and Musical Instruments"[1] that go into the details of the relative importance of these different processes, and also gives numbers that quantify the decay rates of different partials for steel and nylon strings on a guitar. In short, their conclusion is that for nylon strings, the resistance to bending is the most important, while for steel strings, the air resistance is the most important. For neither string type is the transfer to the guitar body important in order to understand how the string decays, while in both cases it is of course important for how the guitar works, since it is due to this process that we are able to hear the sound of the string at all.

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So the expression that represents all these processes is as follows. It's quite a big expression, but it contains it all, including decay:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left(\frac{1}{k(1-k)} \right) \sin(n\pi k) \exp(-\alpha_n t) \sin\left(\frac{n\pi}{L}x\right) \cos\left(\frac{cn\pi}{L}t\right)$$

The expression can be used to compute a value of u for any value of x and t where x is a point on the string, and t is a point in time. For instance, you can compute $u(20, 5)$ which would give you distance (such as 0.25 millimeters), which is the distance the string is from its equilibrium position at a location 20 centimeters from the end of the string, at 5 seconds after the string was plucked. You can also plot $u(x,t)$, for instance if you plot the function for all values of x but with t constant at 5 seconds, you get a picture of the shape of the string 5 seconds after it was plucked. If instead, you plot the function for all values of t but with x constant at 20 centimeters, you get a picture of how the point 20 centimeters from the end of the string moves with time. The other symbols in the expression represent

- L — This is the length of the string, for the guitar it would be 65 cm or so.
- k — This is the point of the string where it is plucked, represented as a fraction of the whole string. If the string is plucked at the mid point, $k = 0.5$, while $k = 0.9$ or $k = 0.1$ represents the string being plucked close to one or the other end of the string.
- n — This is the partial, so that $n = 1$ for the fundamental, $n = 2$ for the first overtone (the octave), $n = 3$ for the next overtone (octave + fifth), etc.
- c — This is the ratio of the tension of the string to the mass of the string.

[1] V. Howle and L. N. Trefethen, Eigenvalues and Musical Instruments, *J. Comp. Appl. Math*, Vol. 135, No. 1, pp. 23–40, October 1, 2001, avaliable from `http://csmr.ca.sandia.gov/~vehowle/publications/pubs.html`.

Partial	Decay rate (steel)	Decay rate (nylon)
1 (fundamental	0.1	0.4
2 (octave)	0.13	0.55
3 (etc...)	0.16	0.7
4	0.18	0.9
5	0.21	1.05
6	0.22	1.15
7	0.23	1.30
8	0.25	1.45
9	0.27	1.55
10	0.28	1.65

Table 1: Decay rates of the ten first partials for a guitar equipped with steel or nylon strings. Values taken from V.E. Howle and Lloyd N. Trefethen: "Eigenvalues and Musical Instruments" without permission.

$-\alpha_n$ This is the decay rate. I can't do the math for computing these, but they can be read off the plots in the paper by Howle and Trefethen, particularly from figure 7 and 8.

$u(x,t)$ This is the amplitude of the string at a given point x at a given time t. $u(x,t)$ is measured from the equilibrium position, which is the poisition of the string when it is not moving, i.e. the string is straight. In other words, if the string is not moving, $u(x,t) = 0$ for all values of x and for all values of t. If the displacement of the string from its equilibrium position at a point 25 cm from the end of the string at a time 1.25 seconds after the string is plucked is 0.75 mm, would be represented by $u(25, 1.25) = 0.75mm$. Drawing a graph of $u(x,t)$ will give us a picture of the shape and movement of the string.

$\sum_{n=1}^{\infty}$ This represents the sum from $n = 1$ to $n = \infty$, where ∞ represents infinity. It basically means that the movement of the string is the sum of all the partials. The important insight here is that each partial can be handled separately with respect to both shape of the string (i.e. the dependence of $u_n(x,t)$ on x), the movement of the string (i.e. the dependence of $u_n(x,t)$ on t) and the decay, and the overall shape and movement of the string is obtained by the sum of all the partials.

The different terms in the equation represent :

$\frac{2}{n^2\pi^2}\left(\frac{1}{k(1-k)}\right)\sin(n\pi k)$ This expression determines how much there is of each partial at the outset, i.e. when the string has been plucked. Note that this is the only place in the expression where k appears, so it is the only part of the system that depends on where the string is plucked.

$\sin\left(\frac{n\pi}{L}x\right)$ This is the shape of the string. Notet that this term is always zero at the end points of the string ($x = 0$ and $x = L$). For $n = 1$, a half-wave fits between the ends of the string, for $n = 2$ a whole wave, $n = 3$ one and a half wave, etc., i.e. the overtone series.

$\cos\left(\frac{cn\pi}{L}t\right)$ This is the vibration of the string as function of time.

$\exp(-\alpha_n t)$ This describes the decay of the vibration of the nth partial. Each partial decays with its own decay rate α_n, which explains the phenomenon of attack.

$\exp(-\alpha_n t)\cos\left(\frac{cn\pi}{L}t\right)$ This describes the complete time dependence of the position of the string, including both the vibration and the exponential decay.

From the paper by V.E. Howle and Lloyd N. Trefethen, I have estimated the following values for the decay rates of the overtones to a vibrating string tuned to E (330 Hz), see table 1. These values were read off a graph included in their paper, so they are not very accurate. These values can be used as values for the parameter α_n above. I am not sure that they are exactly equivalent, but they should give the right impression of the difference in the behaviour of the attack and decay between steel and nylon strings.